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Lipschitz Stability with Perturbing Liapunov Functionals

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Abstract—In this paper, we will discuss the notion of Lipschitz and Lipschitz ϕ_0-stability for systems of functional differential equations employing the method of perturbed Liapunov functionals.

Keywords—Lipschitz stability, Uniform Lipschitz stability, Lipschitz ϕ_0-stability, Uniform Lipschitz ϕ_0-stability.

1. INTRODUCTION

Lakshmikantham and Leela [1] introduced the perturbing Liapunov function method which permits us to discuss nonuniform properties of solutions of systems of differential equations under weaker assumptions. This method was considered from other view of many authors (see [2–4]).

Akpan *et al.* [5] introduced ϕ_0-stability for ordinary differential equations. This notion was improved and extended to the systems of functional differential equations.

The main purpose of this paper is to discuss Lipschitz stability [6] and Lipschitz ϕ_0-stability [7] of the system of functional differential equations via perturbing Liapunov functional method of [4].

Let $\Re^n$ be the n-dimensional Euclidean real space, with any convenient norm $\|.\|$, and scalar product $(.,.) \leq \|.\| \, \|.\|$, $\Re^+ = [0, \infty)$, and let $C[\Re^+ \times \Re^n, \Re^n]$ denote the space of continuous mapping $\Re^+ \times \Re^n$ into $\Re^n$. The following definitions will be needed in the sequel.

DEFINITION 1.1. *(See [5].) A proper subset $K \subset \Re^n$ is called a cone if*

(i) $\lambda K \subset K$, $\lambda \geq 0$,
(ii) $K + K \subset K$,
(iii) $\bar{K} = K$,
(iv) $K^\circ \neq \emptyset$,
(v) $K \cap (-K) = 0$,

where $\bar{K}$ and K° denote the closure and interior of K, respectively, and ∂K denotes the boundary of K.

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The order relation on $\Re^n$ induced by the cone K is defined as follows. Let $x, y \in K$, then $x \leq_{K_y} \Longleftrightarrow y - x \in K$, and $x \leq_{K^o y} \Longleftrightarrow y - x \in K^o$.

DEFINITION 1.2. *(See [5].) The set K^* is called the adjoint cone if*

$$K^* = \{\phi \in \Re^n : (\phi, x) \geq 0\}, \qquad \text{for } x \in K,$$

satisfies Properties (i)–(v) of Definition 1.1,

DEFINITION 1.3. *(See [5].) A function $g : D \to \Re^n$, $D \subset \Re^n$ is called quasimonotone relative to the cone K, if $x, y \in D$ and $y - x \in \partial K$, then there exists $\phi_o \in K_o^*$ such that $(\phi_o, y - x) = 0$ and $(\phi_o, g(y) - g(x)) \geq 0$.*

DEFINITION 1.4. *(See [8].) A function $b(r)$ is said to belong to the class $\mathcal{K}$ if $a \in C[\Re^+, \Re^+]$, $b(0) = 0$, and $b(r)$ is strictly monotone increasing in r. Let $\tau_0(x, y) = 0$ for $(x, y) \in S^n(\rho) \times S^m(\rho)$.*

Consider systems of functional differential equations

$$x' = f(t, x_t), \qquad x_{t_0} = \psi, \tag{1.1}$$

where $f \in C[J \times C_0, K]$, $K \subset \Re^n$ is a cone, $J = [t_0, \infty)$, and

$$\wp^n = C[[-r, 0], \Re^n], \qquad C_0 = \{\phi \in \wp^n : \|\phi\|_0 < \rho\}, \qquad \text{and} \qquad \|\phi\| = \max_{-r \leq s \leq 0} \|\phi(s)\|,$$

$C[J \times C_0, K]$ denotes the space of continuous mapping $J \times C_0 \to K$.

For $x_t(s) = x(t + s)$, $-r \leq s \leq 0$, and $x_t(t_0, \psi)$ being a solution of (1.1) with initial values $x_{t_0} = \psi$, define

$$S_0(\rho) = \{x_t \in C_0 : \|x_t\| < \rho\}.$$

Following [1], we define a Liapunov functional $V(t, x_t) \in C[J \times C_0, \Re^n]$ which is Lipschitzian in x_t, and the functional

$$D^+V(t, x_t) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t + h, x_{t+h}) - V(t, x_t)].$$

The first work dedicated to this method was done by Lakshmikantham and Leela [1].

DEFINITION 1.5. *(See [6].) The zero solution of (1.1) is said to be Lipschitz stable if for $\epsilon > 0$ and $t_0 \in \Re^+$, there exist a positive constant $\delta(t_0, \epsilon) > 0$ and $M > 1$ such that*

$$\|\psi\| < \delta \Longrightarrow \|x_t(t_0, \psi)\| \leq M\|\psi\|, \qquad M > 1,$$

where x_t is any solution of (1.1).

In the case of uniform Lipschitz stability, δ is independent of t_0.

DEFINITION 1.6. *(See [7].) The zero solution of (1.1) is said to be Lipschitz ϕ_0-stable if for $\epsilon > 0$, $t_0 \in J$, there exist a positive function $\delta(t_0, \epsilon) > 0$ and $M > 1$ such that for $\phi_0 \in K_0^*$*

$$(\phi_0, x_t^*(t_0, \psi)) \leq (\phi_0, M\psi), \qquad t \geq t_0,$$

provided that $(\phi_0, \psi) < \delta$.

In the case of uniform Lipschitz ϕ_0-stability, δ is independent of t_0.

2. LIPSCHITZ STABILITY

In this section, we discuss the concept of perturbing Liapunov functionals method for the Lipschitz stability property of the system of functional differential equations (1.1).

THEOREM 1. *Let $E \subset \Re^n$ be a compact subset and suppose that there exist two functions $g_1 \in C[\Re^+ \times \Re^+, \Re]$ and $g_2 \in C[\Re^+ \times \Re^+, \Re]$, and let there exist two functionals $V_1(t, x_t) \in C[J \times \bar{E}^c, \Re^+]$, $V_2(t, x_t) \in C[J \times S_0^c(\eta), \Re^+]$, with $V_1(t, 0) = V_2(t, 0) = g_1(t, 0) = g_2(t, 0) = 0$ such that*

(A_1) *$V_1(t, x_t)$ is Lipschitzian in x_t and*

$$D^+V_1(t, x_t) \leq g_1(t, V_1(t, x_t)), \qquad (t, x_t) \in J \times S_0(\rho);$$

(A_2) *$V_2(t, x_t)$ is Lipschitzian in x_t and*

$$a\|x_t\| \leq V_2(t, x_t) \leq b\|x_t\|,$$

where $a, b \in \mathcal{K}$, $(t, x_t) \in (J \times S_0^c(\eta))$;

(A_3) *for each $(t, x_t) \in (J \times S_0^c(\eta))$,*

$$D^+V_1(t, x_t) + D^+V_2(t, x_t) \leq g_2(t, V_1(t, x_t) + V_2(t, x_t));$$

(A_4) *also assume that the zero solution of the scalar differential equation*

$$u' = g_1(t, u), \qquad u(t_0) = u_0 \geq 0, \tag{2.1}$$

is Lipschitz stable, and the zero solution of the scalar differential equation

$$w' = g_2(t, w), \qquad w(t_0) = w_0 \geq 0, \tag{2.2}$$

is uniformly Lipschitz stable.

Then, the zero solution of system (1.1) is Lipschitz stable.

PROOF. From the compactness of E, there exists a ρ such that

$$S(E, \rho_0) = \{x_t \in C_0, d(x_t, E) < \rho_0\} \subset S(\rho),$$

where $d(x, E) = \inf_{y \in E} \|x - y\|$. Suppose that $\alpha \geq \rho$ be given and $\alpha_1 = \alpha_1(t_0, \alpha) = \max(\alpha_0, \alpha^*)$, where

$$\alpha_0 = \max[V_1(t_0, \psi) : \psi_0 \in S(\rho) \cap E^c], \qquad \alpha^* \geq V_1(t, x_t).$$

From our assumption, the zero solution of (2.1) is Lipschitz stable, for $t_0 \in \Re^+$, and there exist $\delta_1 = \delta_2(t_0, \epsilon) > 0$, and $M > 1$ such that

$$u(t, t_0, u_0) < Mu_0, \qquad t \geq t_0, \tag{2.3}$$

provided that $u_0 < \delta_1$, $u(t, t_0, u_0)$ being any solution of (2.1).

Also, since the zero solution of equation (2.2) is uniformly Lipschitz stable, letting $0 < \epsilon < \rho$ and $t_0 \in \Re$, and $t_0 \in \Re^+$, there exist $\delta_2 = \delta_2(\epsilon) > 0$, and $N > 1$ such that

$$w(t, t_0, w_0) < Nw_0, \qquad t \geq t_0, \tag{2.4}$$

provided that $w_0 < \delta_2$, where $w(t, t_0, w_0)$ is any solution of equation (2.2).

Following [9], choosing $u_0 = V_1(t_0, u_0)$, and $\delta_1 = b(\delta) + \delta_2$, as $b(s) \to \infty$ with $s \to \infty$, we can choose $\delta_1 = \delta_1(t_0, \epsilon)$ such that

$$b(\delta_1) > \frac{\delta}{2}, \tag{2.5}$$

since $V_1(t, x_t)$ is continuous and $V_1(t, 0) = 0$,

$$\|\psi\| < \delta \quad \text{and} \quad \|V_1(t_0, \psi)\| < b(\epsilon), \qquad b \in \mathcal{K}. \tag{2.6}$$

Now, to prove that the zero solution of system (1.1) is Lipschitz stable, we must prove that for $\epsilon > 0$, $t_0 \in \Re^+$, there exist $\delta^* > 0$ and $M > 1$ such that

$$\|\psi\| < \delta^* \Longrightarrow \|x_t(t_0, \psi)\| \le M\|\psi\|, \qquad t \ge t_0.$$

Suppose that this is not true, there exist $t_1, t_2 > t_0$ such that for $\|\psi\| < \delta^*$,

$$\begin{aligned} \|x_{t_1}(t_0, \psi)\| &= \epsilon, \\ \|x_{t_2}(t_0, \psi)\| &= \delta^*, \\ \|x_t(t_0, \psi)\| &\le M\|\psi\|, \qquad t \in [t_1, t_2]. \end{aligned} \tag{2.7}$$

Let $\delta^* = \epsilon/2M$, and $b(\delta^*) \le a(\epsilon)$, so that the existence of $V_{2.\eta}$ satisfies Condition (A_2). Setting

$$m(t) = V_1(t, x_t(t_0, \psi)) + V_{2.\eta}(t, x_t(t_0, \psi)), \qquad t \in [t_1, t_0],$$

we get

$$D^+ m(t) \le g_2(t, m(t)), \qquad t \in [t_1, t_2],$$

which yields from Theorem 8.1.2 of [9],

$$V_1(t_2, x_{t_2}(t_0, \psi)) + V_{2.\eta}(t_2, x_{t_2}(t_0, \psi)) \le r_2(t_2, t_1, V_1(t_1, x_{t_1}(t_0, \psi)) + V_{2.\eta}(t_1, x_{t_1}(t_0, \psi))),$$

where $r_2(t_1, t_1, w_0) = w_0, r_2(t_1, t_1, w_0)$ is the maximal solution of (2.2).

Also, we have

$$V_1(t_1, x_{t_1}(t_0, \psi)) \le r_1(t_1, t_0, V_1(t_0, \psi)),$$

where $r_1(t_1, t_0, u_0)$ is the maximal solutions of (2.1).

By (2.5) and (2.6), we have

$$V_1(t_1, x_{t_1}(t_0, \psi)) \le \frac{\delta^*}{2}. \tag{2.8}$$

From (2.4) and (2.7), we get

$$V_{2.\eta}(t_1, x_{t_1}(t_0, \psi)) < \frac{\delta^*}{2}. \tag{2.9}$$

Thus, (2.3), (2.7)–(2.9), and (A_2) yield the following contradiction:

$$\begin{aligned} a(\epsilon) &= a\|x_{t_1}(t_0, \psi)\| \\ &\le V_{2.\eta}(t_1, x_{t_1}(t_0, \psi)) \\ &\le b\|x_{t_2}(t_0, \psi)\| \\ &= b(\delta^*) \\ &\le a(\epsilon). \end{aligned}$$

Thus, the zero solution of (1.1) is Lipschitz stable, and the proof is completed.

3. LIPSCHITZ ϕ_0-STABILITY

In this section, we discuss Lipschitz ϕ_0-stability of the system of functional differential equations (1.1) via the perturbing Liapunov functional method.

THEOREM 2. *Suppose that there exist two functions* $G_1 \in C[\Re^+ \times \Re^+, \Re^n]$ *and* $G_2 \in C[\Re^+ \times \Re^+, \Re^n]$, *and two functionals* $V_1(t, x_t) \in C[J \times \bar{E}^c, K]$, $V_2(t, x_t) \in C[J \times S_0(\rho) \cap S_0^c(\eta), K]$, *with* $V_1(t, 0) = V_2(t, 0) = G_1(t, 0) = G_2(t, 0) = 0$ *such that*

(A_5) *for every* $\eta > 0$, *there exists a function* $V_{2.\eta}(t, x_t) \in C[J \times S(\rho) \times S^c(\eta), \Re^+]$, $V_1(t, x_t)$ *is Lipschitzian in* x_t *and*

$$D^+(\phi_0, V_1(t, x_t)) \le G_1(t, V_1(t, x_t)), \qquad (t, x_t) \in J \times s_0(\rho);$$

(A_6) $V_{2.\eta}(t, x_t)$ *is Lipschitzian in* x_t *and*

$$a(\phi_0, x_t^*) \le (\phi_0, V_{2.\eta}(t, x_t)) \le b(\phi_0, x_t^*),$$

where $a, b \in \mathcal{K}$, $(t, x_t) \in J \times S_0(\rho) \cap S_0^c(\eta)$, *and* $\phi_0 \in K_0^*$;

(A_7) *for each* $\phi_0 \in K_0^*, (t, x_t) \in J \times S_0(\rho) \cap S_0^c(\eta)$,

$$D^+(\phi_0, V_1(t, x_t)) + D^+(\phi_0, V_2(t, x_t)) \le G_2(t, V_1(t, x_t) + V_2(t, x_t));$$

(A_8) *the zero solution of the system*

$$u' = G(t, u), \qquad u(t_0) = u_0 \ge 0, \tag{3.1}$$

is Lipschitz ϕ_0*-stable, and the zero solution of the system*

$$w' = G(t, w), \qquad w(t_0) = w_0 \ge 0, \tag{3.2}$$

is uniformly Lipschitz ϕ_0*-stable.*

Then, the zero solution of (1.1) is Lipschitz ϕ_0*-stable.*

PROOF. Since the zero solution of (3.2) is uniformly Lipschitz ϕ_0-stable, let $0 < \epsilon < \rho$, for $t_0 \in \Re^+$, there exist $\delta_0 = \delta_0(\epsilon) > 0$ and $M > 1$ such that for $\phi_0 \in K_0^*$

$$(\phi_0, r_2(t, t_0, w_0)) \le (\phi_0, M w_0), \qquad t \ge t_0, \tag{3.3}$$

provided that $(\phi_0, w_0) < \delta_0$, where $r_2(t, t_0, w_0)$ is the maximal solution of (3.2).

From the assumption on $b(s)$, there exists $\delta_2 = \delta_2(\epsilon) > 0$ such that

$$a(\delta_2) < \frac{\delta_0}{2}. \tag{3.4}$$

From our assumption, the zero solution of (3.1) is Lipschitz ϕ_0-stable, for $t_0 \in \Re^+$, there exists $\delta_1 = \delta_1(t_0, \epsilon) > 0$, $M > 1$, such that for $\phi_0 \in K_0^*$

$$(\phi_0, r_1(t, t_0, u_0)) \le (\phi_0, M u_0), \qquad t \ge t_0, \tag{3.5}$$

provided that $(\phi_0, u_0) < \delta_1$, $r_1(t, t_0, u_0)$ being the maximal solution of (3.1).

Following [1], choosing $\psi = V_1(t_0, \psi)$, since $V_1(t, x_t)$ is continuous and $V_1(t, 0) = 0$, there exists $\delta_3 > 0$ such that for $\phi_0 \in K_0^*$

$$(\phi_0, \psi) < \delta_3 \Longrightarrow (\phi_0, V_1(t_0, \psi)) < \epsilon, \qquad t \ge t_0. \tag{3.6}$$

Now, to prove that the zero solution of (1.1) is Lipschitz ϕ_0-stable, it must be proved that

$$(\phi_0, \psi) < \delta_3 \Longrightarrow (\phi_0, x_t^*(t_0, \psi)) < (\phi_0, M\psi), \qquad M > 1, \quad t \ge t_0,$$

where x_t^* is the maximal solution of (1.1). Suppose that this is not true, then there exist $t_1, t_2 > t_0$ such that for $(\phi_0, \psi) < \delta_3$,

$$\begin{aligned} &(\phi_0, x_{t_1}^*(t_0, \psi)) = \epsilon, \\ &(\phi_0, x_{t_2}^*(t_0, \psi)) < \delta_3, \\ &(\phi_0, x_t^*(t_0, \psi)) \le (\phi_0, M\psi), \qquad t \in [t_1, t_2]. \end{aligned} \tag{3.7}$$

Let $\delta_3 = \eta/2$ and $a(\delta_3) \le a(\epsilon)$, so that Condition ($A_6$) is assured. Setting

$$m(t) = V_1(t, x_t(t_0, \psi)) + V_{2.\eta}(t, x_t(t_0, \psi)), \qquad t \in [t_1, t_0],$$

we get for $\phi_0 \in K_0^*$

$$D^+(\phi_0, m(t)) \le G_2(t, m(t)), \qquad t \in [t_1, t_2],$$

which yields from Theorem 1.8.2 of [9]

$$\begin{aligned} &(\phi_0, V_1(t_2, x_{t_2}(t_0, \psi)) + V_{2.\eta}(t_2, x_{t_2}(t_0, \psi))) \\ &\qquad \le (\phi_0, r_2(t_2, t_1, V_1(t_1, x_{t_1}(t_0, \psi)) + V_{2.\eta}(t_1, x_{t_1}(t_0, \psi)))), \end{aligned}$$

where $r_2(t_1, t_0, w_0)$ is the maximal solution of (3.2) with $r_2(t_1, t_0, w_0) = w_0$.

Also, we have for $\phi_0 \in K_0^*$

$$(\phi_0, V_1(t_1, x_{t_1}(t_0, \psi))) \le (\phi_0, r_1(t_1, t_0, V_1(t_0, \psi))),$$

where $r_1(t_1, t_0, u_0)$ is the maximal solution of (3.1).

By (3.5) and (3.6), we have

$$(\phi_0, V_1(t_1, x_{t_1}^*(t_0, \psi))) \le \frac{\delta_3}{2}. \tag{3.8}$$

From (3.4) and (3.7), we get

$$(\phi_0, V_{2.\eta}(t_1, x_{t_1}^*(t_0, \psi))) < \frac{\delta_3}{2}. \tag{3.9}$$

Thus, (3.3), (3.7)–(3.9), and (A_6) yield the following contradiction:

$$\begin{aligned} a(\epsilon) &= a\left(\phi_0, x_{t_1}^*(t_0, \psi)\right) \\ &\le \left(\phi_0, V_{2.\eta}\left(t_1, x_{t_1}^*(t_0, \psi)\right)\right) \\ &\le b\left(\phi_0, x_{t_2}^*(t_0, \psi)\right) \\ &= b\left(\delta_3\right) \\ &\le a(\epsilon). \end{aligned}$$

Thus, the zero solution of (1.1) is Lipschitz ϕ_0-stable, and the proof is completed.

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